

Repeated Root Constacyclic Codes of Length mp^s over

$$\mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$$

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December 3, 2012

Abstract

We give the structure of λ -constacyclic codes of length $p^s m$ over $R = \mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$ with $\lambda \in \mathbb{F}_{p^r}^*$. We also give the structure of λ -constacyclic codes of length $p^s m$ with $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$, where $\alpha_1, \alpha_2 \neq 0$ and study the self-duality of these codes.

1 Introduction

Codes over rings were introduced by Blake [4] as a generalization of codes over fields. It has been shown [21] that these codes are related to some of the best known non-linear codes, such as the Kerdock, Preparata and Goethals codes. Subsequently, finite rings such as finite chain rings [7, 10, 28] and finite principal rings [14, 16] have been used to construct codes. Codes over the polynomial residue rings $\frac{\mathbb{F}_{p^r}[u]}{u^e} = \mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$ were introduced by Bachoc [2]. Many properties such as the code structure, minimum distance, self-duality and decoding have been studied by Bonnetcaze and Udaya [5], Dougherty et al. [11], and Gulliver and Harada [19, 20]. Codes over these rings have been used in the construction of codes for DNA computing by Siap et al. [30], and more recently by Abualrub and Siap [1] and the authors [17]. These rings are a special case of finite chain rings.

Simple root constacyclic codes over finite chain rings are well known [10, 16]. It was proven by Sălăgean [29] that if $(n, p) \neq 1$, cyclic repeated root codes of length n over finite chain rings are not principal ideals. This is also true for negacyclic codes if p is odd. However, little is known about repeated root constacyclic codes over finite chain rings, or even over polynomial residue rings. Recently, the structure of some constacyclic codes over polynomial residue rings of length p^s was given by Jitman and Udomkanavich [23]. Dinh and Nguyen also studied

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some classes of repeated root constacyclic codes over polynomial rings of even characteristic and constacyclic codes of length p^s over $\mathbb{F}_{p^r} + u\mathbb{F}_{p^r}$ [9]. In the first part of this paper, we give the structure of λ -constacyclic codes of length $p^s m$ over $\mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$, where $\lambda \in \mathbb{F}_{p^r}^*$. Furthermore, we prove that some of these λ -constacyclic codes are equivalent to cyclic codes. In the second part of this paper we deal with λ -constacyclic codes of length $p^s m$ over $\mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$ with $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$, where $\alpha_1, \alpha_2 \neq 0$. In addition, the self-duality of these codes is considered. This extends some results in [9].

The remainder of this paper is organized as follows. Section 2 gives some preliminary results concerning finite chain rings. In Section 3, we study λ -constacyclic codes of length $p^s m$ over $R = \mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$ when λ is a p^s power in \mathbb{F}_{p^r} . The equivalence of these codes to cyclic codes over R is established. In Section 4, we study λ -constacyclic codes of length $p^s m$ over $R = \mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$, where $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$ such that $\alpha_1, \alpha_2 \neq 0$. It is proven that these codes are principally generated, and the self-duality of these codes is considered.

2 Preliminaries

Recall that a finite chain ring is a finite local, principal commutative ring R with $1 \neq 0$ such that its ideals are ordered by inclusion

$$(0) \subsetneq \langle u^e \rangle \subsetneq \langle u^{e-1} \rangle \subsetneq \dots \subsetneq \langle u \rangle \subsetneq R.$$

The residue field of the finite chain ring R is $\mathbb{F}_{p^r} = R/\langle u \rangle$. The polynomial residue ring over \mathbb{F}_{p^r} with nilpotency index e is the finite chain ring $R = \frac{\mathbb{F}_{p^r}[u]}{u^e} = \mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \dots + u^{e-1}\mathbb{F}_{p^r}$, where $u^e = 0$. This ring has nilpotency index e and a unique maximal ideal $\langle u \rangle$. An element $v \in R$ is uniquely expressed as

$$r = v_0 + uv_1 + \dots + u^{e-1}v_{e-1},$$

with $r_i \in \mathbb{F}_{p^r}$. The units of R are of the form $v = v_0 + uv_1 + \dots + u^{e-1}v_{e-1}$ with $v_0 \neq 0$. Hence if the group of units is denoted as R^* we have $R^* = R \setminus \langle u \rangle$.

A code \mathcal{C} of length n over R is a subset of R . If the code is a submodule we say that the code is linear. For a given unit $\lambda \in R$, a code \mathcal{C} is said to be constacyclic, or more generally, λ -constacyclic, if $(\lambda c_{n-1}, c_0, c_1, \dots, c_{n-2}) \in \mathcal{C}$ whenever $(c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$. For example, cyclic and negacyclic codes correspond to $\lambda = 1$ and -1 , respectively. It is well known that the λ -constacyclic codes over a finite chain ring R correspond to ideals in $R[x]/\langle x^n - \lambda \rangle$, which in this paper is denoted as \mathcal{R} . The dual code \mathcal{C}^\perp of \mathcal{C} is defined as

$$\mathcal{C}^\perp = \{v \in R^n \mid [v, w] = 0 \text{ for all } w \in \mathcal{C}\}. \quad (1)$$

The dual of a λ -constacyclic code is a λ^{-1} -constacyclic code.

For a polynomial $f(x) = a_0 + a_1x + \dots + a_rx^r$ with $a_0 \neq 0$ and degree r (hence $a_r \neq 0$), the reciprocal of f is the polynomial

$$f^*(x) = x^r f(x^{-1}) = a_r + a_{r-1}x + \dots + a_0x^r. \quad (2)$$

If a polynomial f is equal to its reciprocal, then f is called self-reciprocal. We can easily verify the following equalities

$$(f^*)^* = f \text{ and } (fg)^* = f^*g^*. \quad (3)$$

Let S be a nonempty set of R , then the annihilator of S denoted by $\text{ann}(S)$ is the set

$$\text{ann}(S) = \{f | fg = 0 \text{ for all } g \in S\}.$$

Lemma 2.1 ([9, Proposition 3.4]) *Let R be a commutative ring and λ a unit in R such that $\lambda^2 = 1$. If \mathcal{C} is a λ -constacyclic code over R , then the dual code \mathcal{C}^\perp of \mathcal{C} is the ideal $\text{ann}^*(\mathcal{C})$ where $\text{ann}^*(\mathcal{C}) = \{f^*(x) | f(x)^*g(x) = 0 \text{ for all } g(x) \in \mathcal{C}\}$.*

If $\mathcal{C} = \mathcal{C}^\perp$, we say that the code is self-dual. A code \mathcal{C} and its dual satisfy the following

$$|\mathcal{C}||\mathcal{C}^\perp| = p^{ren} = |R|^n \text{ and } (\mathcal{C}^\perp)^\perp = \mathcal{C}. \quad (4)$$

Remark 2.2 From (4), there exists a self-dual code of length n over R if and only if en is even.

Definition 2.3 A polynomial f in $R[x]$ is called regular if $\overline{f} \neq 0$. f is called primary if the ideal $\langle f \rangle$ is primary, and f is called basic irreducible if \overline{f} is irreducible in $F_{p^r}[x]$. Two polynomials f and g in $R[x]$ are called coprime if

$$R[x] = \langle f \rangle + \langle g \rangle.$$

Lemma 2.4 ([26, Theorem XIII. 11]) *Let f be a regular polynomial in $R[x]$. Then $f = \alpha g_1 \dots g_r$, where α is a unit and g_1, \dots, g_r are regular primary coprime polynomials. Moreover, g_1, \dots, g_r are unique in the sense that if $f = \alpha g_1 \dots g_r = \beta h_1 \dots h_s$, where α, β are units and g_i and h_i are regular primary coprime polynomials, then $r = s$, and after renumbering $\langle g_i \rangle = \langle h_i \rangle$, $1 \leq i \leq n$.*

Several weights over rings can be defined. The homogenous weight is defined [12] as the following generalization of the Lee weight

- (i) $\forall x \in R \setminus \langle u^{e-1} \rangle$, then $w(x) = p^{r(e-2)}(p^r - 1)$;
- (ii) $\forall x \in \langle u^{e-1} \rangle \setminus \{0\}$, then $w(x) = p^{r(e-1)}$;
- (iii) 0 otherwise.

Throughout this paper, the notation $a \equiv \square \pmod b$ denotes that the integer a is a residue quadratic modulo b .

3 λ -Constacyclic Codes of Length $p^s m$ with λ in \mathbb{F}_{p^r}

Let λ be in $\mathbb{F}_{p^r}^*$. In this section, we give the structure of λ -constacyclic codes of length mp^s over R . For this, we require the following lemma.

Lemma 3.1 *If $f(x) \in R[x]$ is a basic irreducible polynomial, then $f(x)$ is a primary polynomial.*

Proof. Assume that $f(x)$ is basic irreducible and $g(x)h(x) \in \langle f(x) \rangle$. Then $\overline{f}(x)$ is irreducible in $K[x]$, so that $(\overline{f}(x), \overline{g}(x)) = 1$ or $\overline{f}(x)$. If $(\overline{f}(x), \overline{g}(x)) = 1$, then f and g are coprime, and there exist f_1 and g_1 in $R[x]$ such that $1 = f(x)f_1(x) + g(x)g_1(x)$. Hence $h(x) = f(x)h(x)f_1(x) + g(x)h(x)g_1(x)$. Since $g(x)h(x) \in \langle f(x) \rangle$, it follows that $h(x) \in \langle f(x) \rangle$. If $(\overline{f}(x), \overline{g}(x)) = \overline{f}(x)$, then there exist $f_1(x), g_1(x) \in R[x]$ such that $g(x) = f(x)f_1 + u^i g_1(x)$ for some positive integer $i < e$. Then for $k > i$, we have $g^k \in \langle f(x) \rangle$, and thus $f(x)$ is a primary polynomial. \square

Remark 3.2 *Let m be an integer such that $\gcd(p, m) = 1$, and λ_0 in $\mathbb{F}_{p^r}^*$. Then from [16], the polynomial $x^m - \lambda_0$ factors uniquely as a product of monic basic irreducible pairwise coprime polynomials over R , and there is a one-to-one correspondence between the set of monic irreducible divisors in \mathbb{F}_{p^r} and the basic irreducible polynomials. Since \mathbb{F}_{p^r} is a subring of R and the decomposition of $x^m - \lambda_0$ is unique in R , the polynomials f_i are in \mathbb{F}_{p^r} .*

Lemma 3.3 *Let α be a primitive element of \mathbb{F}_{p^r} , and $\lambda = \alpha^i$ for $i \leq p^r - 1$. Then the following holds:*

- (i) $x^n = \lambda$ has a solution in \mathbb{F}_{p^r} if and only if $(n, p^r - 1) | i$;
- (ii) if $n = 2m$ with m an odd integer and p an odd prime power, then $x^n = -1$ has a solution in $\mathbb{F}_{p^r}^*$ if and only if $-1 \equiv \square \pmod{p^r}$.

Proof. For Part (i), assume that $x^n = \lambda$ has a solution in \mathbb{F}_{p^r} . Then this solution is equal to $\gamma = \alpha^j$ for some j and satisfies $(\alpha^j)^n = \alpha^i$. This is equivalent to $\alpha^{nj-i} = 1$. Since the order of α is $p^r - 1$, then $(p^r - 1) | nj - i \Leftrightarrow nj - r(p^r - 1) = i$ for some integer r . This gives that $(n, p^r - 1) | i$.

Assuming the existence of a solution α^i of $x^n = -1$, then from Part (i) we have that $(n, p^r - 1) | i$. If n is even and p is odd then $(n, p^r - 1)$ is even, hence i is even. This gives that $-1 \equiv \square \pmod{p^r}$. Conversely, assume that $-1 \equiv \square \pmod{p^r}$. Then there exists an even $i = 2i'$ such that $-1 = \alpha^i$. Since $n = 2m$ is odd, $(-1)^m = -1 = \alpha^{2mi'} = (\alpha^{i'})^n$, and hence there exists a solution of $x^n + 1 = 0$ in \mathbb{F}_{p^r} . \square

Remark 3.4 Lemma 3.3 gives that every $\alpha \in \mathbb{F}_{p^r}$ is a p^s power of an element in \mathbb{F}_{p^r} .

Lemma 3.5 Let λ be a non-zero element of \mathbb{F}_{p^r} , and $n = mp^s$ be an integer such that $\gcd(m, p) = 1$. Then $x^n - \lambda$ has a unique decomposition over R given by

$$x^n - \lambda = f_1^{p^s} \dots f_l^{p^s}, \quad (5)$$

where the f_i are irreducible polynomials coprime in $\mathbb{F}_{p^r}[x]$ which are divisors of $x^m - \lambda_0$, where $\lambda = \lambda_0^{p^s}$.

Proof. From Lemma 3.3 Part (i), we have that for any $\lambda \in \mathbb{F}_{p^r}^*$ there exists λ_0 such that $\lambda = \lambda_0^{p^s}$. That is because $\gcd(p^s, p^s - 1) = 1$. Hence we have that $(x^m - \lambda_0)^{p^s} = x^{mp^s} - \lambda$ because $p \mid \binom{p^s}{i}$ for $1 \leq i \leq p^s$, and so $x^{mp^s} - \lambda = (x^m - \lambda_0)^{p^s}$. From Remark 3.2, $x^m - \lambda_0$ has a unique decomposition into irreducible polynomials over \mathbb{F}_{p^r} given by

$$x^m - \lambda_0 = f_1 \dots f_l. \quad (6)$$

We need to prove that $x^n - \lambda = f_1^{p^s} \dots f_l^{p^s}$ is unique. Assume that $x^n - \lambda = g_1^{\alpha_1} \dots g_l^{\alpha_l}$ is a decomposition into powers of basic irreducible polynomials. From Lemma 3.1, we have that the basic irreducible polynomials are primary, hence the power of a basic irreducible polynomial is also a primary polynomial. Then from Lemma 2.4, the decomposition (5) is unique. \square

Proposition 3.6 With the previous notation, the primary ideals of \mathcal{R} are $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle f_i^j \rangle$, $\langle f_i^j, u^t \rangle$, with $1 \leq j \leq p^s$, $1 \leq t < e$ and $1 \leq i \leq l$.

Proof. From Lemma 3.5 we have $x^n - \lambda = f_1^{p^s} \dots f_l^{p^s}$. Let $\mu : R[x] \mapsto \frac{\mathbb{F}_{p^r}[x]}{\langle x^n - \lambda \rangle}$ be the canonical homomorphism. By Lemma 3.5, we have that the factorization of $x^n - \lambda$ over $R[x]$ is the same as that over $\mathbb{F}_{p^r}[x]$ and is unique. This gives that the kernel of μ is the ideal $\langle x^n - \lambda, u \rangle$. Hence from [32, Theorem 3.9.14], the primary ideals of \mathcal{R} are the preimages of the primary ideals of $\mathbb{F}_{p^r}[x]/x^n - \lambda$. It is well known [15, Theorem 3.10] that the primary ideals of this last ring are the ideals $\langle f_i^j \rangle$, $1 \leq j \leq p^s$ and $1 \leq i \leq l$. Hence the primary ideals of \mathcal{R} are $\langle f_i^j, u^t \rangle$. \square

Theorem 3.7 Let $n = mp^s$ such that $(m, p) = 1$, $\lambda \in \mathbb{F}_{p^r}$, and $x^n - \lambda = \prod f_i^{p^s}$ be the unique factorization into a product of irreducible polynomials in \mathbb{F}_{p^r} . Then the λ -constacyclic codes of length $p^s m$ over R are the ideals generated by $\langle F_0 | u F_1 | \dots | u^{e-1} F_{e-1} \rangle$, where F_i for $0 \leq s_i \leq F_i$ are the monic polynomial divisors of $x^n - \lambda$, and such that $F_i | F_0$ for all $i \leq e-1$.

Proof. Let \mathcal{C} be a constacyclic code in $R[x]$ so that \mathcal{C} is an ideal of \mathcal{R} . Since \mathcal{R} is Noetherian, from the Lasker-Noether decomposition Theorem [32, p. 209], any ideal in \mathcal{R} has a representation as a product of primary ideals. From Proposition 3.6, we have that the primary ideals of \mathcal{R} are $\langle f_i^j, u^t \rangle$. Hence an ideal I of \mathcal{R} is of the form

$$I = \prod_{l=1}^r \langle f_i^j, u^t \rangle. \quad (7)$$

Expanding the product in (7), each ideal in \mathcal{R} is generated by

$$\langle F_0 | u F_1 | u^2 F_2 | \dots | u^{e-1} F_{e-1} \rangle,$$

where the F_i for $0 \leq s_i \leq e-1$ are the monic polynomial divisors of $x^n - \lambda$. \square

Corollary 3.8 *The cyclic codes of length p^s over R are of the form*

$$\langle (x-1)^{j_0} | u(x-1)^{j_1} | \dots | u^{e-1}(x-1)^{j_{e-1}} \rangle$$

Proof. Follows from Theorem 4.3 and the decomposition $x^{p^s} - 1 = (x-1)^{p^s}$. \square

Remark 3.9 *For $e = 2$, the results of Corollary 3.8 were also proven in [8, Theorem 5.4] and [23, Theorem 3.4].*

In the following, we prove that the constacyclic codes considered in this section are equivalent to cyclic codes.

Proposition 3.10 *Let $n = mp^s$ such that $(m, p) = 1$, and $\lambda \in \mathbb{F}_{p^r}$ be a $p^s m$ power in \mathbb{F}_{p^r} . Then a λ -constacyclic code over R of length n is equivalent to a cyclic code of length n .*

Proof. Let $\lambda \in \mathbb{F}_r$ be a $p^s m$ power of $\lambda_0 \in \mathbb{F}_{p^r}^*$, and define

$$\begin{aligned} \phi: R[x]/(x^n - 1) &\longrightarrow R[x]/(x^n - \lambda) \\ f(x) &\longmapsto \phi(f(x)) = f(\delta_0^{-1}x) \end{aligned}$$

It is obvious that ϕ is a ring homomorphism. Hence we only need prove that ϕ is a one-to-one map and it is an isometry for the homogenous weight. For this, let $f(x)$ and $g(x)$ be polynomials in $\mathbb{F}_q[x]$ such that $f(x) \equiv g(x) \pmod{x^n - 1}$. This is equivalent to the existence of $h(x) \in R[x]$ such that $f(x) - g(x) = h(x)(x^n - 1)$. This equality is true if and only if $f(\lambda_0^{-1}x) - g(\lambda_0^{-1}x) = h(\lambda^{-1}x)[(\lambda^{-1}x)^n - 1]$ is true. We have that $h(\lambda^{-1}x)[\lambda^{-n}x^n - 1] = \lambda^{-n}h(\lambda^{-1}x)[x^n - \delta^n] = \delta^{-n}h(\delta^{-1}x)[x^n - \lambda]$, so for $f, g \in \mathbb{F}_q[x]/(x^n - 1)$

$$\phi(f(x)) = \phi(g(x)) \iff g(x) = f(x).$$

Then ϕ is well defined and one-to-one, and hence is a ring isomorphism. Now we need to prove that ψ is an isometry according to a homogeneous weight over R . Let $w(\cdot)$ be a homogeneous weight over R and let $f(x) = a_0 + a_1x + \dots + a_nx^n$ be a codeword in $R[x]/(x^n - 1)$. Then $\psi(f(x)) = a_0 + a_1\delta^{-1}x + a_2\delta^{-1}x^2 + \dots + \delta^{-1}x^n$. Since δ is a unit, it must be that $w(\delta^{-i}a_i) = w(a_i)$, and hence $w(\psi(f(x))) = w(f(x))$. Then $\psi(A)$ is an ideal of $R[x]/(x^n - \lambda_0)$ and if B is an ideal of $R[x]/(x^n - \lambda_0)$, $\psi^{-1}(B)$ is an ideal of $R[x]/(x^n - 1)$. Since the map ψ is a ring isomorphism which is a homogeneous isometry, the codes A and $\psi(A)$ are equivalent by the result in [13]. \square

Theorem 3.11 *Let $n = mp^s$ be an odd integer such that $(m, p) = 1$, and $\lambda \in \mathbb{F}_{p^r}^*$ such that there exists $\delta \in \mathbb{F}_q^*$ and $\delta^m = \lambda$. Then the following hold:*

- (i) $\pm\lambda$ -constacyclic codes of length mp^s over R are equivalent to cyclic codes over R ;
- (ii) if $p^r \equiv 1 \pmod{4}$ and $\delta = \beta^2$ in \mathbb{F}_q , then $\pm\lambda$ -constacyclic codes of length $2mp^s$ over R are equivalent to cyclic codes over R .

Proof.

- i) Let $\lambda \in \mathbb{F}_q^*$ such that there exists $\delta \in \mathbb{F}_q^*$ and $\delta^m = \lambda$. Then there exists $\alpha \in \mathbb{F}_q^*$ such that $\alpha^{p^s} = \delta$, so that $\lambda = \delta^m = \alpha^{mp^s}$. Since mp^s is odd, we obtain that $-\lambda = (-\delta)^m = (-\alpha)^{mp^s}$, and the result follows by Proposition 3.10.
- ii) Let $\lambda \in \mathbb{F}_q^*$ such that there exists $\beta \in \mathbb{F}_q^*$ and $\beta^{2m} = \lambda$. Then there exists $\rho \in \mathbb{F}_q^*$ such that $\rho^{p^s} = \beta$ and $\lambda = \beta^{2m} = \rho^{2mp^s}$. Thus ρ is a $2mp^s$ -th root of λ in \mathbb{F}_q . If $q \equiv 1 \pmod{4}$, by Lemma 3.3 there exists $\xi \in \mathbb{F}_q$ such that $\xi^2 = -1$. Then $-\lambda = (-1)^{mp^s} \beta^{2m} = \xi^{2mp^s} \rho^{2mp^s} = (\xi\rho)^{2mp^s}$, and $\xi\rho$ is a $2mp^s$ -th root of $-\lambda$ in \mathbb{F}_q . The result then follows from Proposition 3.10.

\square

Example 3.12 *For $\lambda \in \mathbb{F}_{p^r}^*$, the λ -constacyclic codes over R of length p^s are equivalent to cyclic codes.*

Let m be an odd integer and $p^r \equiv 1 \pmod{4}$ an odd prime power. From Part (ii) of Theorem 3.11, the negacyclic codes of length $2mp^s$ are equivalent to cyclic codes of length $2mp^s$ over R .

4 Constacyclic Codes of Length $p^s m$

Let $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$ be a unit of R such that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and there exists $\alpha \in \mathbb{F}_{p^r}$ which satisfies $\alpha_1 = \alpha_0^{p^s}$. This is a generalization of the λ -constacyclic codes of Type 1* given by Dinh [9], thus the approach follows that in [9]. Conditions on self-duality are also given. We first prove the following lemma.

Lemma 4.1 *Let p be an odd prime, and $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$ such that $\alpha_1 = \alpha_0^{p^s} \in \mathbb{F}_{p^r}^*$. Then in \mathcal{R} we have $(\alpha_0^{-1}x^m - 1)^{p^s} = u\rho$ where ρ is a unit in R and $\alpha_0^{-1}x^m - 1$ is nilpotent in \mathcal{R} with nilpotency index ep^s . Furthermore, any $f \in R[x]$ which is coprime to $\alpha_0^{-1}x^m - 1$ is invertible in $\mathcal{R}[x]$.*

Proof. In $\mathcal{R}(\alpha)$ we have

$$\begin{aligned} (\alpha_0^{-1}x^m - 1)^{p^s} &= (\alpha_0^{-1}x)^{mp^s} + (-1)^{p^s} + \sum_{i=1}^{p^s-1} \binom{p^s}{i} (\alpha_0^{-1}x)^{p^s-i} \\ &= \alpha_0^{-p^s} x^{mp^s} - 1 \\ &= \alpha_1^{-1}(\alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}) - 1 \\ &= \alpha_1^{-1}(u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}) \\ &= u(\alpha_1^{-1}\alpha_2 + \dots + u^{e-2}\alpha_1^{-1}\alpha_{e-1}). \end{aligned}$$

Hence $\rho = \alpha_1^{-1}(\alpha_2 + \dots + u^{e-2}\alpha_{e-1})$ is a unit, since $\alpha_2 \neq 0$. Thus in \mathcal{R} we have $\langle \alpha_0^{-1}x^m - 1 \rangle = \langle u \rangle$, so the nilpotency index of $\alpha_0^{-1}x^m - 1$ is equal to ep^s . For the second part of the lemma, since f and $\alpha_0^{-1}x^m - 1$ are coprime, there exists $g, h \in R[x]$ such that $(\alpha_0^{-1}x^m - 1)g(x) + h(x)f(x) = 1$, or equivalently $h(x)f(x) = 1 - Y$, where $Y = (\alpha_0^{-1}x^m - 1)g(x)$. As we have already proven that $(\alpha_0^{-1}x^m - 1)^{p^s} = 0$ in $\mathcal{R}[x]$, we obtain that $Y^{p^s} = 0$. Hence $1 = 1 - Y^{p^s} = (1 - Y)(1 + Y + \dots + Y^{p^s-1})$, which means that $h(x)f(x)$ is invertible and so f is invertible in \mathcal{R} . \square

Lemma 4.2 *With the previous notation, in $\mathcal{R}[x]$ we have $\langle f_i^{p^s} \rangle = \langle f_i^{p^s+k} \rangle$ for any i and k , where k is a positive integer.*

Proof. We have that f_i and $\tilde{f}_i = \frac{\alpha^{-1}x^m-1}{f_i}$ are coprime, so that f_i^k and $(\tilde{f}_i)^{p^s}$ are also coprime. Then there exist $g, h \in R[x]$ such that $f_i^k g + \tilde{f}_i^{p^s} h = 1$. Hence $f_i^{k+p^s} g = (1 - \tilde{f}_i^{p^s} h) f_i^{p^s} = f_i^{p^s} - (x^n + 1)^{p^s} h = f_i^{p^s} h$. Thus in \mathcal{R} we have that $\langle f_i^{p^s} \rangle = \langle f_i^{p^s+k} \rangle$. \square

Theorem 4.3 *Let $n = mp^s$ such that $\gcd(m, p) = 1$, and let $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$ such that $\alpha_1 = \alpha_0^{p^s} \in \mathbb{F}_{p^r}^*$ and $\alpha_2 \neq 0$. Then $x^n - \alpha_1 = \prod f_i^{p^s}$ factors uniquely as the product of irreducible polynomials in \mathbb{F}_{p^r} and \mathcal{R} is a principal ideal ring whose ideals are generated by*

$$\langle \prod f_i^{s_i} \rangle, \text{ where } 0 \leq s_i \leq p^s e. \quad (8)$$

Moreover we have that

$$|\mathcal{C}| = p^{re(n - \sum_{i \in I} s_i \deg f_i)}.$$

Proof. Since $\alpha_0^{p^s} = \alpha_1$, from Lemma 3.5 we have that $x^n - \alpha_1 = f_1^{p^s} \dots f_l^{p^s}$ factors uniquely as the product of irreducible polynomials in \mathbb{F}_{p^r} . Let \mathcal{C} be a λ -constacyclic code in R so that \mathcal{C} is an ideal of \mathcal{R} . Further, let \mathcal{C}_u be the set of codewords of \mathcal{C} reduced modulo u . Then \mathcal{C}_u is an ideal of $\frac{\mathbb{F}_{p^r}[x]}{x^n - \alpha_1}$, and hence $\mathcal{C}_u = \langle \prod f_i^{l_i} \rangle$, where $l_i \leq p^s$. Then for $c(x) \in \mathcal{C}$, there exists g and h in \mathcal{R} such that $c(x) = h(x) \prod f_i^{l_i}(x) + ug(x)$. However, since in \mathcal{R}_λ we have from Lemma 4.1 that $u \in \langle (\alpha_0^{-1}x^m - \alpha_1)^{p^s} \rangle = \prod f_i^{p^s}(x)$, then $\mathcal{C} \subset \langle \prod f_i^{l_i}(x) \rangle$. Let s_i be the largest power of f_i such that $\mathcal{C} \subset \langle \prod f_i^{s_i}(x) \rangle$. Then from Proposition 3.5, we have $0 \leq s_i \leq p^s$. If $c(x) \in \mathcal{C}$, then $c(x) = c'(x) \prod f_i^{s_i}(x)$ with $c'(x) \in \mathcal{R}$. Since the s_i are maximal, we have that $\gcd(c'(x), \prod f_i^{s_i}(x)) = 1$, and hence $\gcd(c'(x), \alpha_0^{-1}x^m - 1) = 1$. By Lemma 4.1, $c'(x)$ is invertible, so that $\prod f_i^{s_i}(x) = c'(x)c^{-1}(x) \in \mathcal{C}$. Thus $\mathcal{C} = \prod f_i^{s_i}(x)$ with $s_i \leq p^s e$, and hence $|\mathcal{C}| = |R|^{n - \sum s_i \deg(f_i)} = p^{re(n - \sum s_i \deg(f_i))}$. \square

Since the f_i are pairwise coprime, the ideals $\langle \prod_{i \in I} f_i^{s_i} \rangle$, $0 \leq s_i \leq p^s e$, are distinct. Hence the number of λ -constacyclic codes is given in the following lemma.

Corollary 4.4 *With the previous notation, the number of λ -constacyclic codes of length mp^s is equal to*

$$(p^s e + 1)^l,$$

where l is the number of the p^r -cyclotomic classes modulo m .

Example 4.5 *If $m = 1$, then $x^{p^s} - \alpha_1 = (x - \alpha_0)^{p^s}$. Hence in this case \mathcal{R} is a chain ring with the following ideals*

$$\langle 0 \rangle = \langle (x - \alpha_0)^{ep^s} \rangle \subsetneq \langle (x - \alpha_0)^{ep^s - 1} \rangle \dots \subsetneq \langle (x - \alpha_0) \rangle \subsetneq \langle 1 \rangle = \langle \mathcal{R} \rangle.$$

4.1 Self-Dual λ -Constacyclic Codes

In this section, we study the self-duality of λ -constacyclic codes.

Theorem 4.6 *Let $n = mp^s$ such that $\gcd(m, p) = 1$, and $\lambda = \alpha_1 + u\alpha_2 + \dots + u^{e-1}\alpha_{e-1}$ such that $\alpha_1 = \alpha_0^{p^s} \in \mathbb{F}_{p^r}^*$ and $\alpha_2 \neq 0$. Let \mathcal{C} be a λ -constacyclic code of length $p^s m$ generated by the polynomial $\prod_i f_i^{s_i}$ with $s_i \leq p^s e$. Then the dual \mathcal{C}^\perp of \mathcal{C} is a λ^{-1} -constacyclic code of length mp^s . If $\lambda^2 = 1$, then \mathcal{C}^\perp is a λ -constacyclic code generated by*

$$\prod_i (f_i^*)^{p^s e - s_i}.$$

Proof. We know that the dual of a λ -constacyclic code \mathcal{C} is a λ^{-1} -constacyclic code. Hence \mathcal{C} can be self-dual if and only if $\lambda^2 = 1$. From (4), the cardinality is $|\mathcal{C}^\perp| = \frac{|R|^n}{p^{re(n - \sum s_i \deg(f_i))}} = p^{re \sum s_i \deg(f_i)}$. Now denote $\hat{\mathcal{C}} = \langle \prod_i (f_i)^{p^s e - s_i} \rangle$. Then we need only prove that $\hat{\mathcal{C}}^* \subset \mathcal{C}^\perp$ have the same cardinality. We have the following equality in \mathcal{R}

$$\prod_i (f_i)^{p^s e - s_i} \prod_i (f_i)^{s_i} = \prod_i (f_i)^{p^s e} = (x^m - \alpha_1)^{p^s e} = 0.$$

Hence it follows from Lemma 2.1 that $\hat{\mathcal{C}}^* \subset \text{ann}^*(\mathcal{C}) = \mathcal{C}^\perp$. Since $\hat{\mathcal{C}}$ is an ideal of \mathcal{R} , from Theorem 4.3 we have that $|(\hat{\mathcal{C}})^*| = |\hat{\mathcal{C}}| = p^{re(\sum_{i \in I} s_i \deg f_i)}$. This has the same cardinality as \mathcal{C}^\perp , since $\mathcal{C}^\perp = \frac{|R^n|}{p^{re - (n - \sum_{i \in I} s_i \deg f_i)}} = p^{re(\sum_{i \in I} s_i \deg f_i)}$ from (4). \square

Now we give condition on the existence of λ -constacyclic self-dual codes over R . For this we need the following decomposition.

Denote the self-reciprocal factors in the factorization of $x^m - \alpha_0$ by g_1, \dots, g_k , and the remaining factors grouped in pairs by $h_1, h_1^*, \dots, h_t, h_t^*$. Hence $l = k + 2t$, and we have the following factorization:

$$\begin{aligned} x^n - \alpha_1 &= (x^m - \alpha_0)^{p^s} = g_1^{p^s}(x) \dots g_k^{p^s}(x) \\ &\quad \times h_1^{p^s}(x) h_1^{*p^s}(x) \dots h_t^{p^s}(x) h_t^{*p^s}(x). \end{aligned} \quad (9)$$

Theorem 4.7 *Let p be an odd prime with the notation above, λ such that $\lambda^2 = 1$, and e odd. Then there exists a self-dual λ -constacyclic code of length mp^s over R if and only if m is even and there are no g_i (self-reciprocal polynomials) in the factorization of $x^{mp^s} - \alpha_1$ given in (9). Further, a self-dual λ -constacyclic code \mathcal{C} is generated by a polynomial of the form*

$$h_1^{b_1}(x) h_1^{*p^s e - b_1}(x) \dots h_t^{b_t}(x) h_t^{*p^s e - b_t}(x). \quad (10)$$

Proof. From Remark 2.2, we must have that m is even since it was assumed that e and p are odd. If there exists a λ -constacyclic self-dual code \mathcal{C} of length $n = mp^s$ over R , then from (8) it is generated by $A(x) = \prod f_i^{k_i}$, where the f_i are factors of $x^m - \alpha_0$. From (9), we can write

$$A(x) = g_1^{a_1}(x) \dots g_k^{a_k}(x) h_1^{b_1}(x) h_1^{*c_1}(x) \dots h_t^{b_t}(x) h_t^{*c_t}(x),$$

where $0 \leq a_i \leq p^s e$ for $1 \leq i \leq k$, and $0 \leq b_j \leq p^s e$ and $0 \leq c_j \leq p^s e$ for $1 \leq j \leq t$. Let $\langle B(x) \rangle = \mathcal{C}^\perp$. Then from Theorem 4.6 we obtain

$$\begin{aligned} B(x) &= g_1^{p^s e - a_1}(x) \dots g_s^{p^s e - a_k}(x) \\ &\quad \times h_1^{*p^s e - b_1}(x) h_1^{p^s e - c_1}(x) \dots h_t^{*p^s e - b_t}(x) h_t^{p^s e - c_t}(x). \end{aligned}$$

Since \mathcal{C} is self-dual, by equating factors of $A(x)$ and $B(x)$, the powers of the factors of $A(x)$ and $B(x)$ must satisfy $a_i = p^s e - a_i$ for $1 \leq i \leq k$, and $c_j = p^s e - b_j$ for $1 \leq j \leq t$.

Equivalently, $p^s e = 2a_i$ for $1 \leq i \leq k$, and $c_j = p^s e - b_j$ for $1 \leq j \leq t$. Since pe is odd, the last equalities are possible if and only if there are no g_i in the factorization of $x^{mp^s} - \lambda$ and $c_j = p^s e - b_j$ for $1 \leq j \leq t$, i.e, $k = 0$ in (9) and $c_j = p^s e - b_j$ for $1 \leq j \leq t$. Hence a λ -constacyclic self-dual code is generated by

$$h_1^{b_1}(x)h_1^{*p^s e - b_1}(x) \dots h_t^{b_t}(x)h_t^{*p^s e - b_t}(x).$$

□

The condition given in Theorem 4.7 that there are no irreducible factors of $x^m - \alpha_0$ which are self-reciprocal is equivalent to $p^{ri} \not\equiv -1 \pmod{m}$ for all positive integers i [18]. Hence we obtain the following corollary.

Corollary 4.8 *With the previous notation, assume that λ is such that $\lambda^2 = 1$, p is an odd prime, e is odd and m is an even integer. Then non-trivial λ -constacyclic self-dual codes of length n over R exist if and only if $(p^r)^i \not\equiv -1 \pmod{m}$ for all positive integers i .*

Corollary 4.9 *With the previous notation, assume that λ is such that $\lambda^2 = 1$, e is odd, m an even integer, and p an odd prime such that $p \equiv 5 \pmod{8}$ or $p \equiv 3 \pmod{8}$. Then no λ -constacyclic code exists over R .*

Proof. From [27, Theorem 6], if $p \equiv 5 \pmod{8}$ or $p \equiv 3 \pmod{8}$, then there exists $i > 0$ such that $(p^r)^i \not\equiv -1 \pmod{m}$. The result then follows from Corollary 4.8. □

Example 4.10 *If e is odd. Then there is no λ -constacyclic self-dual code of length $5^s \cdot 6$, $s \geq 1$ over $\mathbb{F}_5 + u\mathbb{F}_5 + \dots + u^{e-2}\mathbb{F}_5$.*

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